

Arc-connectivity and super arc-connectivity of mixed Cayley digraph

Yuhu Liu* Jixiang Meng

College of Mathematics and System Sciences, Xinjiang University
Urumqi, Xinjiang, 830046, P.R.China

Abstract

A digraph $X = (V, E)$ is max- λ , if $\lambda(X) = \delta(X)$. A digraph X is super- λ if every minimum cut of X is either the set of inarcs of some vertex or the set of outarcs of some vertex. In this paper, we'll prove that for all but a few exceptions, the strongly connected *mixed Cayley digraphs* are max- λ and super- λ .

Keywords: Mixed Cayley digraph, arc-connectivity, λ -atom, λ -superatom.

1 Introduction

Let $X = (V, E)$ be a digraph, where V is a finite set and E is an irreflexive relation on V , thus E is a set of ordered pairs $(u, v) \in V \times V$ such that $u \neq v$, the elements of V are called the *vertices* or *nodes* of X and the elements of E are called the *arcs* of X , arc (u, v) is said to be an *inarc* of v and an *outarc* of u . If u is a vertex of X , then the *outdegree* of u in X is the number $d_X^+(u)$ of arcs of X originating at u and the *indegree* of u in X is the number $d_X^-(u)$ of arcs of X terminating at u . The minimum outdegree of X is $\delta^+(X) = \min\{d_X^+(u) \mid u \in V\}$, the minimum indegree of X is $\delta^-(X) = \min\{d_X^-(u) \mid u \in V\}$, we denote by $\delta(X)$ the minimum of $\delta^+(X)$ and $\delta^-(X)$.

An *arc-disconnecting set* of X is a subset W of E such that $X \setminus W = (V, E \setminus W)$ is not strongly connected. An arc disconnecting set is *minimal* if no proper subset of W is an arc disconnecting set of X and is a *minimum arc disconnecting set* if no other arc disconnecting set has smaller cardinality than W . The *arc connectivity* $\lambda(X)$ of a nontrivial digraph X is the cardinality of a minimum arc disconnecting set of X .

The *positive arc neighborhood* of a subset A of V is the set $\omega_X^+(A)$ of all arcs which initiate at a vertex of A and terminate at a vertex of $V \setminus A$. The *negative neighborhood* of subset A of V is the set $\omega_X^-(A)$ of all arcs which initiate in $V \setminus A$ and terminate in A . Clearly $\omega_X^-(A) = \omega_X^+(V \setminus A)$. Arc neighborhoods of proper, nonempty subsets of V , often called cuts, are clearly arc disconnecting sets.

*Corresponding author.

E-mail address: xjliu@163.com (Y.H.Liu), mjx@xju.edu.cn (J.Meng).

The research is supported by NSFC ,FUKT and SPUX.

An *arc fragment* of X is a proper, nonempty subset of V whose positive or negative arc neighborhood has cardinality $\lambda(X)$.

We define a digraph X to be *super arc-connected*, or more simply, *super- λ* , if every minimum cut of X is either the set of inarcs of some vertex or the set of outarcs of some vertex.

Let $X = (V, E)$ be a strongly connected digraph. An arc fragment of least possible cardinality is called a λ -*atom* of X and a nontrivial arc fragment of least possible cardinality is called a λ -*superatom* of X .

Definition 1.1 The *reverse* digraph of digraph $X = (V, E)$ is the digraph $X^{(r)} = (V, \{(v, u) \mid (u, v) \in E\})$, digraph $X = (V, E)$ is *symmetric* if $E = E^{(r)}$ and is *antisymmetric* if $E \cap E^{(r)} = \emptyset$.

Definition 1.2 If G is a group and S is a subset of $G \setminus \{1_G\}$, where 1_G is the identity of G . We define the *Cayley digraph* $\text{Cay}(G, S)$ to be the digraph with vertices the elements of group G and arcs all pairs of the form $(g, s \cdot g)$ with $g \in G$ and $s \in S$. We define a *Cayley graph* to be a symmetric Cayley digraph. It should be clear that a *Cayley digraph* $\text{Cay}(G, S)$ is symmetric if and only if the inverse of every element of S is again in S .

Definition 1.3 Let G be a group, $T_0, T_1 \subseteq G$, the *Bi-Cayley digraph* of G with respect to T_0 and T_1 is defined as the bipartite digraph with vertex set $G \times \{0, 1\}$ and arc set $\{((g, 0), (t_0 \cdot g, 1)), ((t_1 \cdot g, 1), (g, 0)) \mid g \in G, t_0 \in T_0, t_1 \in T_1\}$, denoted by $BD(G, T_0, T_1)$.

J.X.Meng gives the definition of *mixed Cayley digraph*. In order to be convenient in this paper, we narrate it by another way.

Definition 1.4 Let G be a finite group, $S_0, S_1 \subseteq G \setminus \{1_G\}$, $T_0, T_1 \subseteq G$. Define the *mixed Cayley digraph*

$MD = G(G, S_0, S_1, T_0, T_1) = \text{Cay}(G \times \{0\}, S_0) \cup \text{Cay}(G \times \{1\}, S_1) \cup BD(G, T_0, T_1)$ as follows:

- 1) $V(MD) = G \times \{0, 1\}$, and let $X_0 = G \times \{0\}$, $X_1 = G \times \{1\}$.
- 2) $((g, i), (s_i \cdot g, i)) \in E(MD)$, $g \in G, s_i \in S_i$, for $i = 0, 1$.
- 3) $((g, 0), (t_0 \cdot g, 1)) \in E(MD)$, $((t_1 \cdot g, 1), (g, 0)) \in E(MD)$ for $t_0 \in T_0, t_1 \in T_1$ and $g \in G$.

So far, the research on the connectivity of the Cayley graph is mainly focused on vertex connectivity, results on this subject are referred to [7, 8, 9]. The research on the Bi-Cayley graph is primarily focused on its isomorphisms[3], few results, if any, are known on graphic properties of Bi-Cayley graphs. The results of Mixed Cayley graph are few. In [2], Chen and Meng point out that the Mixed Cayley graph also has high connectivity. In this paper, we study the arc-connectivity of strongly connected Mixed Cayley digraph, and we will prove that the strongly connected Mixed Cayley digraphs

are max- λ and super- λ but a few exceptions.

We denote by $Aut(X)$ the automorphism group of X . The graph X is said to be *vertex transitive* if $Aut(X)$ acts transitively on $V(X)$, and to be *edge transitive* if $Aut(X)$ acts transitively on $E(X)$. It is proved that these two kinds of graphs usually have high connectivity. For instance, connected vertex transitive graphs have maximum edge connectivity[4], and connected edge transitive graphs have maximum vertex connectivity[8].

For $a \in G$, the right multiplication $R'(a): g \rightarrow ga, g \in G$, is clearly an automorphism of any Cayley digraph of G . Let $R'(G) = \{R'(a): a \in G\}$, then $R'(G)$ is a subgroup of *the automorphism group* of any Cayley digraph. In following proposition, we'll prove that $R(G) = \{R(a)|R(a): (g, i) \rightarrow (ga, i), \text{ for } a, g \in G \text{ and } i=0,1\}$ is also a subgroup of *the automorphism group* of any *mixed Cayley digraph*.

Proposition 1.5 Let $X = MD(G, S_0, S_1, T_0, T_1)$, then

- (1) $R(G) \leq Aut(X)$, thus $Aut(X)$ acts transitively both on X_0 and X_1 .
- (2) $d_X^+((g, 0)) = |T_0| + |S_0|$, $d_X^-((g, 0)) = |T_1| + |S_0|$,
 $d_X^+((g, 1)) = |T_1| + |S_1|$, $d_X^-((g, 1)) = |T_0| + |S_1|$, for any $g \in G$.

proof. (1) $((g_1, i), (g_2, i)) \in E(X) \Leftrightarrow g_2 = s_i g_1$ for some $s_i \in S_i \Leftrightarrow g_2 a = s_i g_1 a \Leftrightarrow ((g_1 a, i), (g_2 a, i)) \in E(X) \Leftrightarrow R(a)((g_1, i), (g_2, i)) \in E(X)$ for $i = 0, 1$.

$((g_1, 0), (g_2, 1)) \in E(X) \Leftrightarrow g_2 = t_0 g_1$ for some $t_0 \in T_0 \Leftrightarrow g_2 a = t_0 g_1 a \Leftrightarrow ((g_1 a, 0), (g_2 a, 1)) \in E(X) \Leftrightarrow R(a)((g_1, 0), (g_2, 1)) \in E(X)$.

$((g_2, 1), (g_1, 0)) \in E(X) \Leftrightarrow g_2 = t_1 g_1$ for some $t_1 \in T_1 \Leftrightarrow g_2 a = t_1 g_1 a \Leftrightarrow ((g_2 a, 1), (g_1 a, 0)) \in E(X) \Leftrightarrow R(a)((g_2, 1), (g_1, 0)) \in E(X)$.

So for any $a \in G$, $R(a)$ is an automorphism of the mixed Cayley digraph X , thus $R(G) \leq Aut(X)$, and since $R(g_1^{-1} g_2)((g_1, i)) = (g_2, i)$ for any $g_1, g_2 \in G$, $Aut(X)$ acts transitively both on X_0 and X_1 .

- (2) $N^+((g, 0)) = \{\{T_0 g\} \times \{1\}\} \cup \{\{S_0 g\} \times \{0\}\}$,
 $N^-((g, 0)) = \{\{T_1 g\} \times \{1\}\} \cup \{\{S_0^{-1} g\} \times \{0\}\}$,
 $N^+((g, 1)) = \{\{T_1^{-1} g\} \times \{0\}\} \cup \{\{S_1 g\} \times \{1\}\}$
 $N^-((g, 1)) = \{\{T_0^{-1} g\} \times \{0\}\} \cup \{\{S_1^{-1} g\} \times \{1\}\}$,

so we can get

$$\begin{aligned} d_X^+((g, 0)) &= |T_0| + |S_0|, \quad d_X^-((g, 0)) = |T_1| + |S_0^{-1}| = |T_1| + |S_0|, \\ d_X^+((g, 1)) &= |T_1^{-1}| + |S_1| = |T_1| + |S_1|, \quad d_X^-((g, 1)) = |T_0^{-1}| + |S_1^{-1}| = |T_0| + |S_1|. \quad \square \end{aligned}$$

2 Many results we need in this paper

Proposition 2.1[8] Let $X = (V, E)$ be a strongly connected digraph and let A and B be positive(respectively, negative) arc fragments of X such that $A \not\subseteq B$ and $B \not\subseteq A$. If $A \cap B \neq \emptyset$ and $A \cup B \neq V$, then each of the sets $A \cap B$, $A \cup B$, $A \setminus B$ and $B \setminus A$ is an positive (respectively, negative) arc fragments of X . \square

Corollary 2.2 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be strongly connected mixed Cayley digraph, If $\lambda(X) < \delta(X)$, distinct positive(respectively, negative) λ -atoms are vertex disjoint. \square

An *imprimitive block* for a group Φ of permutations of a set T is a proper, nontrivial subset A of T such that if $\varphi \in \Phi$ then either $\varphi(A) = A$ or $\varphi(A) \cap A = \emptyset$.

Theorem 2.3[8] Let $X = (V, E)$ be a graph or digraph and let Y be the subgraph or subdigraph induced by an imprimitive block A of X . Then

1. If X is vertex-transitive then so is Y .
2. If X is a strongly connected arc-transitive digraph or a connected edge-transitive graph and A is a proper subset of V , then A is an independent subset of X .
3. If $X = Cay(G, S)$ and A contains the identity of G , then A is a subgroup of G . \square

Theorem 2.4[8] If $X = (V, E)$ is a strongly connected digraph, but not super- λ and has $\delta(X) > 2$, then distinct positive(respectively, negative) λ -superatoms of X are vertex disjoint. \square

Theorem 2.5[8] Every strongly connected vertex-transitive digraph X satisfies $\lambda(X) = \delta(X)$. \square

3 Arc-connectivity of the mixed Cayley digraph

In this section, we'll prove that for all but a few exceptions, the mixed Cayley digraph is max- λ . Clearly, if either T_0 or T_1 is empty, $X = MD(G, S_0, S_1, T_0, T_1)$ isn't strongly connected, so in following paper, we suppose that $T_0 \neq \emptyset$ and $T_1 \neq \emptyset$.

Proposition 3.1 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be a strongly connected mixed Cayley digraph and A be a λ -atom. If $\lambda(X) < \delta(X)$, then

- (1) $Y = X[A]$ is a strongly connected subdigraph of X .
- (2) $|A| \geq \delta(X) + 1$.

Proof. Without loss of generality, we suppose A is a positive λ -atom.

- (1) If Y is not strongly connected, there exists a proper subset B of A such that

$$\omega_Y^+(B) = \emptyset, \text{ so } \omega_X^+(B) \subseteq \omega_X^+(A),$$

thus

$$|\omega_X^+(A)| = |\omega_X^+(B)| \text{ and } |B| < |A|.$$

It's a contradiction.

- (2) Because $\lambda(X) = |\omega_X^+(A)| \geq |A|(\delta(X) - |A| + 1)$, if $2 \leq |A| \leq \delta(X)$, we can verify that

$$|A|(\delta(X) - |A| + 1) \geq \delta(X),$$

thus when $2 \leq |A| \leq \delta(X)$, $\lambda(X) \geq \delta(X)$, it is a contradiction. \square

Lemma 3.2 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be a strongly connected mixed Cayley digraph and A be a λ -atom. If $\lambda(X) < \delta(X)$, then $|A \cap X_i| \geq 2$ where $X_i = \{ (g, i) \mid g \in G \}$, for $i = 0, 1$.

Proof. Without loss of generality, suppose A is a positive λ -atom.

Claim 1 $A_i = A \cap X_i \neq \emptyset$ for $i = 0, 1$.

If $A_0 = \emptyset$ or $A_1 = \emptyset$, then

$$\lambda(X) = \omega_X^+(A) \geq \min\{|A||T_0|, |A||T_1|\} \geq |A|,$$

thus by proposition 3.1, $\lambda(X) \geq \delta(X) + 1$, it is a contradiction.

Claim 2 $|A_i| = |A \cap X_i| \geq 2$, for $i = 0, 1$.

Suppose $|A_0| = 1$, then

$$\begin{aligned} \lambda(X) &= |\omega_X^+(A)| = \sum_{v \in A} d_X^+(v) - \sum_{v \in A} d_{X[A]}^+(v) = \\ &= \sum_{v \in A_0} d_X^+(v) + \sum_{v \in A_1} d_X^+(v) - \sum_{v \in A} d_{X[A]}^+(v). \end{aligned}$$

Because there are at most $|T_0| + |T_1|$ arcs between A_0 and A_1 ,

$$\lambda(X) \geq |T_0| + |S_0| + |T_1^{-1}||A_1| - (|T_0| + |T_1|) = |S_0| + (|A_1| - 1)|T_1|.$$

Since X is strongly connected and $\lambda(X) < \delta(X)$, we have $\delta(X) \geq 2$.

By proposition 3.1 $|A_1| = |A| - |A_0| = |A| - 1 \geq \delta(X)$, thus

$$\lambda(X) \geq |S_0| + |T_1| = d_X^-(g, 0) \geq \delta(X).$$

It is a contradiction. \square

Lemma 3.3 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be a strongly connected mixed Cayley digraph and $A = A_0 \cup A_1$ be a λ -atom, where $A_i = A \cap X_i = H_i \times \{i\}$, and $H_i \subseteq G$ for $i = 0, 1$. Set $Y_i = X[A_i]$ be the subdigraph of X induced by A_i for $i = 0, 1$. If $\lambda(X) < \delta(X)$, then

- (1) $Aut(Y_i)$ acts transitively on A_i for $i = 0, 1$.
- (2) If A_i contains $(1_G, i)$, then H_i is a subgroup of X for $i = 0, 1$.

Proof. (1) By lemma 3.2, A_i is nontrivial, for any $(g_1, i), (g_2, i) \in A_i$, by proposition 1.5, $R(g_2^{-1}g_1) \in R(G) \leq Aut(X)$. And it's easy to verify that $R(g_2^{-1}g_1)(A)$ is also a λ -atom, so $R(g_2^{-1}g_1)(A) = A$. Using proposition 1.5(1) and theorem 2.5, we can deduce that $R(g_2^{-1}g_1)(A_i) = A_i$ for $i = 0, 1$. So the restriction of $R(g_2^{-1}g_1)$ on A_i induces an automorphism of Y_i , which maps (g_1, i) to (g_2, i) . Because (g_1, i) and (g_2, i) are two arbitrary vertices of A_i , $Aut(Y_i)$ acts transitively on A_i for $i = 0, 1$.

(2) By lemma 3.2, $|A_i| \geq 2$. Then for any arbitrary vertex $(g, i) \in A_i$, $R(g^{-1})((g, i)) = (1_G, i)$, so $R(g^{-1})(A) = A$, it means that $Ag^{-1} = A$, so $A_i g^{-1} = A_i$, thus we get that $hg^{-1} \in H_i$, for any $h, g \in H_i$, so H_i is a subgroup of G for $i = 0, 1$. \square

Lemma 3.4 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be a strongly connected mixed Cayley digraph, and A be a λ -atom of X , let $A_i = A \cap X_i$, then if $\lambda(X) < \delta(X)$, we have that

- (1) $V(X)$ is a disjoint union of distinct positive(respectively, negative) λ -atoms of X .
- (2) $|A_0| = |A_1|$.

Proof. (1) Without loss of generality, set A be a positive λ -atom, by proposition 1.5, $Aut(X)$ acts transitively both on X_0 and X_1 . Because $\lambda(X) < \delta(X)$, from theorem 2.5, X isn't vertex transitive. Thus X has exactly two orbits X_0 and X_1 , by lemma 3.2, $|A_i| \geq 2$, so at least one vertex of X_i lines in A respectively. So every vertex of X lines in a positive λ -atom. By corollary 2.2, $V(X)$ is a disjoint union of distinct positive λ -atoms.

(2) Let $V(X) = \cup_{i=1}^k \varphi_i(A)$, where $\varphi_i \in Aut(X)$ such that $\varphi_i(A) \cap \varphi_j(A) \neq \emptyset$ if and only if $i = j$, then $X_i = \cup_{i=1}^k \varphi_i(A_i)$. Since $|X_0| = |X_1|$, we have $|A_0| = |A_1|$. \square

Lemma 3.5 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be a strongly connected mixed Cayley digraph with $\lambda(X) < \delta(X)$ and $A = A_0 \cup A_1$ be a λ -atom, where $A_i = A \cap X_i = H_i \times \{i\}$ and $H_i \subseteq G$ for $i = 0, 1$. Then

(1) If $(1_G, 0) \in A_0$, then $H_1 = t_i H_0$ for some $t_i \in T_i$, furthermore,

$$\begin{aligned} X_0 &= \cup_{j=1}^k (H_0 g_j) \times \{0\}, \\ X_1 &= \cup_{j=1}^k (t_i H_0 g_j) \times \{1\}, \end{aligned}$$

where $R(g_j)(A) \cap R(g_l)(A) \neq \emptyset$ if and only if $j = l$ for $1 \leq j, l \leq k$.

(2) If $(1_G, 1) \in A_1$, then $H_0 = t_i^{-1} H_1$ for some $t_i \in T_i$, furthermore,

$$\begin{aligned} X_0 &= \cup_{j=1}^k (t_i^{-1} H_1 g_j) \times \{0\}, \\ X_1 &= \cup_{j=1}^k (H_1 g_j) \times \{1\}, \end{aligned}$$

where $R(g_j)(A) \cap R(g_l)(A) \neq \emptyset$ if and only if $j = l$ for $1 \leq j, l \leq k$.

Proof. (1) Since $(1_G, 0) \in A_0$ and $X[A]$ is strongly connected by proposition 3.1, there must exist at least an element $t_i \in T_i$ such that $t_i \in H_1$. If $(1_G, 0) \in A_0$, $H_0 \leq G$.

Then for any $h_0 \in H_0$, $R(h_0)(A) = A$, since $H_0 h_0 = H_0$.

Thus for any $h_0 \in H_0$, $H_1 h_0 = H_1$, so $H_1 H_0 = H_1$.

And because $t_i \in H_1$ and $|H_0| = |H_1|$, we have that $H_1 = t_i H_0$.

Since $H_0 \leq G$, we get that $G = \cup_{j=1}^k (H_0 g_j)$, where $g_1 = 1_G$ and $H_0 g_j \cap H_0 g_l \neq \emptyset$ if and only if $j = l$ for $1 \leq j, l \leq k$. Therefore,

$$V(X) = \cup_{j=1}^k R(g_j)(A).$$

So $X_0 = \cup_{j=1}^k R(g_j)(A_0) = \cup_{j=1}^k (H_0 g_j) \times \{0\}$,

$$X_1 = \cup_{j=1}^k R(g_j)(A_1) = \cup_{j=1}^k (H_1 g_j) \times \{1\} = \cup_{j=1}^k (t_i H_0 g_j) \times \{1\}.$$

(2) It is similar to (1). \square

proposition 3.6 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be a strongly connected mixed Cayley digraph with $\lambda(X) < \delta(X)$, and let A be a λ -atom of X , and let $Y = X[A]$, then $Aut(Y)$ acts transitively both on A_0 and A_1 , where $A_i = A \cap X_i$ for $i = 0, 1$.

Proof. It is clearly true from proposition 1.5, corollary 2.2 and lemma 3.3. \square

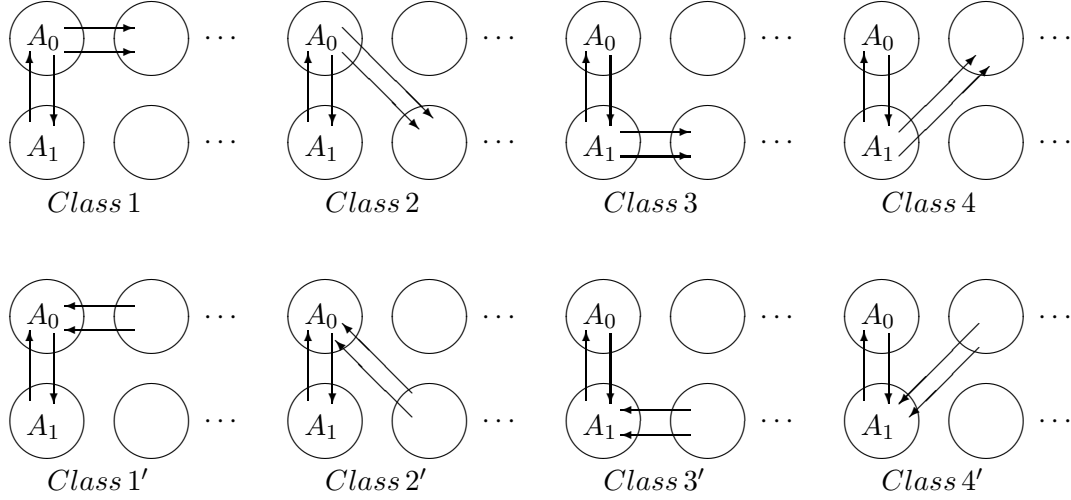
Set $H = Y \setminus \{E(Y_0) \cup E(Y_1)\}$ where $Y_i = X[A_i]$, $i = 0, 1$, then from lemma 3.3 and proposition 3.6, we can get

$d_H^+((g, 0)), d_H^-((g, 0)), d_H^+((g, 1)), d_H^-((g, 1)), d_{Y_i}^+(g, i)$ and $d_{Y_i}^-(g, i)$ are constant respectively. Furthermore,

$$d_H^+((g, 0)) = d_H^-((g, 1)), d_H^-((g, 0)) = d_H^+((g, 1)) \text{ and } d_{Y_i}^+(g, i) = d_{Y_i}^-(g, i).$$

So we set $d_H^+((g, 0)) = d_H^-((g, 1)) = p$, $d_H^-((g, 0)) = d_H^+((g, 1)) = q$ and Y_i is r_i regular digraph.

If X is a strongly connected mixed Cayley digraph with $\lambda(X) < \delta(X)$, from lemma 3.4, $V(X)$ is the union of distinct positive (respectively, negative) λ -atoms of X . Set A is a λ -atom of X and $A_i = A \cap X_i$, for $i=0, 1$. Now we introduce a class of digraphs consisting of the following eight classes of digraphs, denoted by Γ ,



where $|A_0| = |A_1| < \delta(X)$ and Class 1 satisfies

$$|S_0| - r_0 = 1, |T_0| - p = 0, |S_1| - r_1 = 0 \text{ and } |T_1| - q = 0.$$

The Class 2 satisfies

$$|S_0| - r_0 = 0, |T_0| - p = 1, |S_1| - r_1 = 0 \text{ and } |T_1| - q = 0.$$

The Class 3 satisfies

$$|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 1 \text{ and } |T_1| - q = 0.$$

The Class 4 satisfies

$$|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 0 \text{ and } |T_1| - q = 1.$$

the Class 1' satisfies

$$|S_0| - r_0 = 1, |T_0| - p = 0, |S_1| - r_1 = 0 \text{ and } |T_1| - q = 0.$$

The Class 2' satisfies

$$|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 0 \text{ and } |T_1| - q = 1.$$

The Class 3' satisfies

$$|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 1 \text{ and } |T_1| - q = 0.$$

The Class 4' satisfies

$$|S_0| - r_0 = 0, |T_0| - p = 1, |S_1| - r_1 = 0 \text{ and } |T_1| - q = 0.$$

Clearly, the Class 1 and the Class 3 are equivalent to the Class 1' and the Class 3' respectively. And we can also easily prove that the Class 2 and the Class 4 are equivalent to the Class 4' and Class 2' respectively.

Theorem 3.7 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be a strongly connected mixed Cayley digraph. Then X is not $\max-\lambda$ if and only if X belongs to the class of digraphs Γ .

Proof. Necessity. If $\lambda(X) < \delta(X)$, by proposition 3.6, we set that

$d_H^+((g, 0)) = d_H^-((g, 1)) = p$, $d_H^-((g, 0)) = d_H^+((g, 1)) = q$, and Y_i is r_i -regular digraph. Let A be a λ -atom.

1. When A is a positive λ -atom, then

$$\lambda(X) = |\omega_X^+(A)| = |A_0|(|S_0| - r_0 + |T_0| - p) + |A_1|(|S_1| - r_1 + |T_1| - q).$$

Since $|A| \geq \delta(X) + 1$ and $|A_0| + |A_1| \geq \delta(X) + 1$, we have $|A_0| = |A_1| > \delta(X)/2$.

So $\lambda(X) = |\omega_X^+(A)| < \delta(X)$ is true only if one of the following conditions holds.

Case 1 $|S_0| - r_0 + |T_0| - p = 1$ and $|S_1| - r_1 + |T_1| - q = 0$.

Subcase 1.1 $|S_0| - r_0 = 1, |T_0| - p = 0, |S_1| - r_1 = 0$ and $|T_1| - q = 0$, clearly, under this subcase X is Class 1.

Subcase 1.2 $|S_0| - r_0 = 0, |T_0| - p = 1, |S_1| - r_1 = 0$ and $|T_1| - q = 0$, clearly, under this subcase X is Class 2.

Case 2 $|S_0| - r_0 + |T_0| - p = 0$ and $|S_1| - r_1 + |T_1| - q = 1$.

Subcase 2.1 $|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 1$ and $|T_1| - q = 0$, clearly, under this subcase X is Class 3.

Subcase 2.2 $|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 0$ and $|T_1| - q = 1$, clearly, under this subcase X is Class 4.

2. When A is a negative λ -atom, then

$$\lambda(X) = |\omega_X^-(A)| = |A_0|(|S_0| - r_0 + |T_1| - q) + |A_1|(|S_1| - r_1 + |T_0| - p).$$

Since $|A| \geq \delta(X) + 1$ and $|A_0| = |A_1|$, we have that $|A_0| = |A_1| > \delta(X)/2$.

So if $\lambda(X) = |\omega_X^-(A)| < \delta(X)$, one of the following conditions holds,

Case 1' $|S_0| - r_0 + |T_1| - q = 1$ and $|S_1| - r_1 + |T_0| - p = 0$.

Subcase 1'.1 $|S_0| - r_0 = 1, |T_1| - q = 0, |S_1| - r_1 = 0$ and $|T_0| - p = 0$, clearly, under this subcase X is Class 1'.

Subcase 1'.2 $|S_0| - r_0 = 0, |T_1| - q = 1, |S_1| - r_1 = 0$ and $|T_0| - p = 0$, clearly, under this subcase X is Class 2'.

Case 2' $|S_0| - r_0 + |T_1| - q = 0$ and $|S_1| - r_1 + |T_0| - p = 1$.

Subcase 2'.1 $|S_0| - r_0 = 0, |T_1| - q = 0, |S_1| - r_1 = 1$ and $|T_0| - p = 0$, clearly, under this subcase X is Class 3'.

Subcase 2'.2 $|S_0| - r_0 = 0, |T_1| - q = 0, |S_1| - r_1 = 0$ and $|T_0| - p = 1$, clearly, under this subcase X is Class 4'.

Sufficiency, it is clearly true. \square

Proposition 3.8 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, X is Class 1 or Class 1' if and only if

(1) There exists a non-empty proper subgroup H of G and S_0 contains an element s_0 such that

- $< S_0 \cup \{1_G\} \setminus \{s_0\} > \leq H$ and $|H| < \delta(X)$, and
(2) There is an element $t_0 \in T_0$ such that
 $G_1 = < S_1 > \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H$ and $t_0^{-1} T_0 \subseteq H$.

Proof. Necessity. Because Class 1 is equivalent to Class 1', without loss of generality, we set X is Class 1, then Assume $(1_G, 0) \in A_0$, by lemma 3.3, $H_0 \leq G$. Let $H = H_0$, then under this situation we can achieve the following results easily,

- (i) $\lambda(X) = |\omega_X^+(A)| = |A_0| = |H_0| = |H| < \delta(X)$, since $|S_0| - r_0 = 1$, $|T_0| - p = 0$, $|S_1| - r_1 = 0$, and $|T_1| - q = 0$,
(ii) $< S_0 \cup \{1_G\} \setminus \{s_0\} > \leq H_0 = H$, since $|S_0| - r_0 = 1$.

By proposition 3.5, $H_1 = t_0 H_0$ for some $t_0 \in T_0$ and $X_1 = \cup_{i=1}^k (t_0 H_0 g_i) \times \{1\}$, where $t_0 H_0 g_i \cap t_0 H_0 g_j \neq \emptyset$ if and only if $i = j$ for $1 \leq i, j \leq k$. Assume that $(1_G, 1) \in (t_0 H_0 g_s) \times \{1\}$, then we can deduce that $t_0 H_0 g_s \leq G$ and $g_s = h_0^{-1} t_0^{-1}$, where $h_0 \in H_0$.

Since $|S_1| - r_1 = 0$, we get $G_1 \leq t_0 H_0 g_s = t_0 H_0 h_0^{-1} t_0^{-1} = t_0 H_0 t_0^{-1} = t_0 H t_0^{-1}$.

Since $|T_0| - p = 0$ and $|T_1| - q = 0$, we have that $T_0 H_0 \subseteq H_1$ and $T_1^{-1} H_1 \subseteq H_0$,

So $T_0 H_0 \subseteq t_0 H_0$ and $T_1^{-1} t_0 H_0 \subseteq H_0$,

it means that

$$t_0^{-1} T_0 \subseteq H_0 = H \text{ and } T_1^{-1} t_0 \subseteq H_0 = H \text{ for some } t_0 \in T_0.$$

Sufficiency, set $A = H \times \{0\} \cup (t_0 H) \times \{1\}$,

because $< S_0 \cup \{1_G\} \setminus \{s_0\} > \leq H$, we can get $|S_0| - r_0 = 1$.

Similarly,

because $G_1 = < S_1 > \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H$ and $t_0^{-1} T_0 \subseteq H$, we can get that

$$|S_1| - r_1 = 0, |T_0| - p = 0 \text{ and } |T_1| - q = 0.$$

So $\lambda(X) \leq |\omega^+(A)| = |H| < \delta(X)$. \square

Analogously, we can achieve the following proposition 3.9, 3.10 and 3.11 easily.

Proposition 3.9 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, X is Class 2 or Class 4' if and only if

- (1) There exists a non-empty proper subgroup H of G such that

$$G_0 = < S_0 > \leq H \text{ and } |H| < \delta(X), \text{ and}$$

- (2) There are two distinct elements $t_0, t'_0 \in T_0$ such that

$$G_1 = < S_1 > \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H,$$

$$t'_0 H \cap t_0 H = \emptyset \text{ and } t_0^{-1} (T_0 \setminus \{t'_0\}) \subseteq H. \square$$

Proposition 3.10 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, X is Class 3 or Class 3' if and only if

- (1) There exists a non-empty proper subgroup H of G and some element $s_1 \in S_1$ such that

$$< S_1 \cup \{1_G\} \setminus \{s_1\} > \leq H \text{ and } |H| < \delta(X), \text{ and}$$

- (2) There is an element $t_1 \in T_1$ such that

$$G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H \text{ and } t_1 T_1^{-1} \subseteq H. \quad \square$$

Proposition 3.11 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, X is Class 4 or Class 2' if and only if

- (1) There exists a non-empty proper subgroup H of G such that
 $G_1 = \langle S_1 \rangle \leq H$ and $|H| < \delta(X)$, and
- (2) There are two distinct elements $t_1 \in T_1$ and $t'_1 \in T_1$ such that
 $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H,$
 $t_1'^{-1} H \cap t_1^{-1} H = \emptyset$ and $t_1(T_1^{-1} \setminus \{t_1'^{-1}\}) \subseteq H. \quad \square$

Put the above propositions together, we get the following theorem.

Theorem 3.12 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be a strongly connected mixed Cayley digraph. Then X is not max- λ if and only if X satisfies one of the following conditions:

Condition 1.

- (1.1) There exists a non-empty proper subgroup H of G and S_0 contains an element s_0 such that
 $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H$ and $|H| < \delta(X)$, and
- (1.2) There is an element $t_0 \in T_0$ such that
 $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H$ and $t_0^{-1} T_0 \subseteq H.$

Condition 2.

- (2.1) There exists a non-empty proper subgroup H of G such that
 $G_0 = \langle S_0 \rangle \leq H$ and $|H| < \delta(X)$, and
- (2.2) There are two distinct elements $t_0, t'_0 \in T_0$ such that
 $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H,$
 $t'_0 H \cap t_0 H = \emptyset$ and $t_0^{-1}(T_0 \setminus \{t'_0\}) \subseteq H.$

Condition 3.

- (3.1) There exists a non-empty proper subgroup H of G and some element $s_1 \in S_1$ such that
 $\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq H$ and $|H| < \delta(X)$, and
- (3.2) There is an element $t_1 \in T_1$ such that
 $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H$ and $t_1 T_1^{-1} \subseteq H.$

Condition 4.

- (4.1) There exists a non-empty proper subgroup H of G such that
 $G_1 = \langle S_1 \rangle \leq H$ and $|H| < \delta(X)$, and
- (4.2) There are two distinct elements $t_1 \in T_1$ and $t'_1 \in T_1$ such that
 $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H,$
 $t_1'^{-1} H \cap t_1^{-1} H = \emptyset$ and $t_1(T_1^{-1} \setminus \{t_1'^{-1}\}) \subseteq H. \quad \square$

So all the strongly connected mixed Cayley digraphs but a few exceptions are max- λ

4 Super arc-connectivity of mixed Cayley digraph

If a digraph isn't max- λ , it is also not super- λ . So in this section, we prove that the mixed Cayley digraph, which is max- λ but not super- λ , is only a few exceptions. A *weak path* of a digraph X is a sequence u_0, \dots, u_r of distinct vertices such that for $i = 1, \dots, r$, either (u_{i-1}, u_i) or (u_i, u_{i-1}) is an arc of X . A directed graph is *weakly connected* if any two vertices can be joined by a weak path. The following proposition is clearly true.

proposition 4.1 If $X = MD(G, S_0, S_1, T_0, T_1)$ is max- λ , but not super- λ , let A be a λ -super atom, then $Y = X[A]$ is weakly connected. \square

Lemma 4.2 If $X = MD(G, S_0, S_1, T_0, T_1)$ is max- λ , but not super- λ , let A be a λ -superatom, If $\delta(X) = 1$, $A \cap X_i \neq \emptyset$.

Proof. By contradiction. Because X is strongly connected, $|T_0| \geq 1$ and $|T_1| \geq 1$. If $\delta(X) = 1$, one of the following conditions holds:

- (1) $|S_0| = 0$, $|T_0| = 1$ or $|T_1| = 1$,
- (2) $|S_1| = 0$, $|T_0| = 1$ or $|T_1| = 1$.

Without loss generality, suppose (1) holds, then $A \not\subseteq X_0$, thus $\lambda(X) \geq |A||T_1| \geq |A| \geq 2$, a contradiction. \square

Lemma 4.3 Let $X = MD(G, S_0, S_1, T_0, T_1)$ is max- λ , but not super- λ , let A be a λ -superatom, then $|A| \geq \delta(X)$.

Proof. Suppose A is a positive λ -superatom. Then

$$\lambda(X) = |\omega_X^+(A)| \geq |A|(\delta - (|A| - 1)) = |A|(\delta - |A| + 1),$$

we can verify that $\lambda(X) > \delta(X)$ when $2 \leq |A| < \delta(X)$, a contradiction. So $|A| \geq \delta(X)$. \square

Let $X = MD(G, S_0, S_1, T_0, T_1) = \text{Cayley}(G \times \{0\}, S_0) \cup \text{Cayley}(G \times \{1\}, S_1) \cup BD(G, T_0, T_1)$ be a strongly connected mixed digraph. There is a class of special mixed Cayley digraph, which is that one of $\text{Cayley}(G \times \{0\}, S_0)$ and $\text{Cayley}(G \times \{1\}, S_1)$ is a union of disjoint directed cycles, and the other is a union of disjoint directed cycles with length two, and $BD(G, T_0, T_1)$ is a union of disjoint directed cycles with length two. The class of special mixed Cayley digraphs is denoted by \mathcal{F} .

Lemma 4.4 Let $X = MD(G, S_0, S_1, T_0, T_1)$ is max- λ but not super- λ , if X is neither a directed cycle nor a cycle and X doesn't belong to \mathcal{F} , then distinct positive(respectively, negative) λ -superatoms of X are vertex disjoint.

Proof. We suppose $\delta(X) \leq 2$, since the lemma is true when $\delta(X) \geq 3$ by theorem 2.4.

Let A and B be two distinct positive λ -superatoms. If $A \cap B \neq \emptyset$, by proposition

2.1, $A \cap B, A \cup B, A \setminus B, B \setminus A$ are arc fragments of X . Because each of $A \cap B, A \setminus B$ and $B \setminus A$ is a proper subset of a λ -superatom, we achieve that

$$|A \cap B| = 1, |A \setminus B| = 1 \text{ and } |B \setminus A| = 1.$$

So assume $A = \{u, v\}, B = \{v, w\}$ with $u \neq w$, thus

$$\begin{aligned} d_{X[A]}^+(u) = d_{X[A]}^-(v) &\leq 1, d_{X[A]}^-(u) = d_{X[A]}^+(v) \leq 1, \\ d_{X[B]}^+(v) = d_{X[B]}^-(w) &\leq 1 \text{ and } d_{X[B]}^-(v) = d_{X[B]}^+(w) \leq 1. \end{aligned}$$

Case 1 $A, B \subseteq X_0$ or $A, B \subseteq X_1$.

$$2 \geq \lambda(X) = |\omega_X^+(A \cup B)| \geq 3\min(|T_0|, |T_1|) \geq 3, \text{ a contradiction.}$$

Case 2 $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset$.

When $\delta(X) = 1$.

Since $A \cap B, A \setminus B$ and $B \setminus A$ are arc-fragments of X , $d_X^+(u) = d_X^+(v) = d_X^+(w) = 1$, so $|T_0| = 1, |T_1| = 1$ and $|S_0| = |S_1| = 0$. And because X is a strongly connected digraph, we can get X is a directed cycle, a contradiction.

When $\delta(X) = 2$, then

$$d_X^+(u) = d_X^+(v) = d_X^+(w) = 2.$$

Because $X[A]$ and $X[B]$ are weakly connected and A, B and $A \cup B$ are arc fragments, we can deduce

$$|T_0| = |T_1| = 2, T_0 = T_1 \text{ and } |S_0| = |S_1| = 0.$$

Because X is strongly connected, X is a cycle, a contradiction.

Case 3 $A \cap X_i \neq \emptyset$ and either $B \subseteq X_0$ or $B \subseteq X_1$ ($B \cap X_i \neq \emptyset$ and $A \subseteq X_0$ or $A \subseteq X_1$).

By lemma 4.2, we can get

$$\delta(X) \geq 2, \text{ so } \delta(X) = 2.$$

Since A, B and $A \cup B$ are arc fragment, We can get that

$$|T_0| = |T_1| = |S_0| = |S_1| = 1 \text{ such that } S_1^{-1} = S_1 \text{ or } S_0^{-1} = S_0 \text{ and } T_0 = T_1,$$

thus X belongs to \mathcal{F} , a contradiction. \square

Lemma 4.5 Let $X = MD(G, S_0, S_1, T_0, T_1)$ is $\max-\lambda$ but not $\text{super}-\lambda$, if X is neither a directed cycle nor a cycle and X doesn't belong to \mathcal{F} . Let A be a λ -superatom of X , then

(1) If $A \subseteq X_i$, let $A = H \times \{i\}, i = 0, 1, H \subseteq G$. And let $Y = X[A]$ be the subgraph of X induced by A , then

(i) $\text{Aut}(Y)$ acts transitively on A , and

(ii) If A contains $(1_G, 0)$ or $(1_G, 1)$, then H is a subgroup of G .

(2) If $A_i = A \cap X_i = H_i \times \{i\} \neq \emptyset$ where $H_i \subseteq G$, let $Y_i = X[A_i]$ be the subgraphs of X induced by A_i , then

(i) $\text{Aut}(Y_i)$ acts transitively on A_i for $i = 0, 1$, and

(ii) If A_i contains $(1_G, i)(i = 0, 1)$, then H_i is a subgroup of G .

Proof. (1) Without loss of generality, suppose $A \subseteq X_0$, then $X[A]$ is the subdigraph of $\text{Cayley}(G \times \{0\}, S_0)$ induced by $H \times \{0\} \subseteq G \times \{0\}$, where $A = H \times \{0\}$.

By lemma 4.4, A is an imprimitive block of $\text{Cayley}(G \times \{0\}, S_0)$.

So by theorem 2.3, we have that (1) holds.

(2) The proof is similar to the proof of lemma 3.3. \square

Lemma 4.6 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be max- λ but not super- λ , if X is neither a directed cycle nor a cycle and X doesn't belong to \mathcal{F} . Let A be a λ -super atom of X with $A_i = A \cap X_i \neq \emptyset$, then

- (1) $V(X)$ is a disjoint union of distinct positive(negative) λ -super atoms, and
- (2) $|A_0| = |A_1|$. \square

The proof of lemma 4.6 is similar to the proof of lemma 3.4.

proposition 4.7 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be max- λ but not super- λ , if X is neither a directed cycle nor a cycle and X doesn't belong to \mathcal{F} . Let A be a λ -super atom of X with $A_i = A \cap X_i \neq \emptyset$. Set $A_i = A \cap X_i = H_i \times \{i\}$, for $i = 0, 1$, where $H_i \subseteq G$, then

- (1) If $(1_G, 0) \in A_0$, then $H_1 = t_0 H_0$ for some $t_0 \in T_0$, furthermore,

$$X_0 = \cup_{i=1}^k (H_0 g_i) \times \{0\},$$

$$X_1 = \cup_{i=1}^k (t_0 H_0 g_i) \times \{1\},$$

where $R(g_i)(A) \cap R(g_j)(A) \neq \emptyset$ if and only if $i = j$ for $1 \leq i, j \leq k$.

- (2) If $(1_G, 1) \in A_1$, then $H_0 = t_1^{-1} H_1$ for some $t_1 \in T_1$, furthermore,

$$X_0 = \cup_{i=1}^k (t_1^{-1} H_1 g_i) \times \{0\},$$

$$X_1 = \cup_{i=1}^k (H_1 g_i) \times \{1\},$$

where $R(g_i)(A) \cap R(g_j)(A) \neq \emptyset$ if and only if $i = j$ for $1 \leq i, j \leq k$. \square

The proof is similar to the lemma 3.5.

We give two classes of digraphs which aren't super- λ . The first of class digraphs consists of the strongly connected mixed Cayley digraphs $X = MD(G, S_0, S_1, T_0, T_1)$ which contain λ -superatoms lining in X_0 or X_1 . This class of digraphs is denoted by \mathcal{G} . The second class of digraphs consists of the strongly connected mixed Cayley digraph $X = MD(G, S_0, S_1, T_0, T_1)$ all of whose λ -superatoms contain at least one vertex of X_0 and X_1 respectively, denoted by \mathcal{L} .

Theorem 4.8 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be max- λ , but not super- λ , if X is neither a directed cycle nor a cycle and X doesn't belong to \mathcal{F} , then X belongs to \mathcal{G} if and only if X satisfies one of the following conditions:

- (1) $|T_0| = 1$ or $|T_1| = 1, 1 \leq |S_0| \leq |S_1|$ and $S_0 \cup \{1_G\} \leq G$.
- (2) $|T_0| = 1$ or $|T_1| = 1, 1 \leq |S_1| \leq |S_0|$ and $S_1 \cup \{1_G\} \leq G$.

Proof. Necessity.

Because $A \subseteq X_0$ or $A \subseteq X_1$, we have $\delta(X) \geq 2$ by the lemma 4.2.

1.1 A is a positive λ -superatom.

If $A \subseteq X_0$ and $(1_G, 0) \in A$, then $A = H \times \{0\}$ and $H \leq G$.

By lemma 4.5, $Y = X[A]$ is a regular digraph, so we set Y is r_0 regular digraph, then

$$\lambda(X) = |\omega_X^+(A)| = |A|(|S_0| + |T_0| - r_0) \geq \delta(X)(|S_0| + |T_0| - r_0).$$

Since $|T_0| \geq 1$ and $\lambda(X) = \delta(X)$, we have that

$$|A| = \delta(X), |S_0| - r_0 = 0 \text{ and } |T_0| = 1.$$

So Y is a $|S_0|$ regular digraph with order $\delta(X)$. Thus

$$|S_0| \leq \delta(X) - 1 = \min\{|S_0| + 1, |S_1| + 1, |S_0| + |T_1|, |S_1| + |T_1|\} - 1 \leq \min\{|S_0| + 1, |S_1| + 1\} - 1,$$

so we can get $|S_0| \leq |S_1|$ and $Y \cong K_{\delta(X)}$. So if X is not super- λ then

$$|T_0| = 1, 1 \leq |S_0| \leq |S_1| \text{ and } S_0 \cup \{1_G\} \leq G.$$

Similarly, if $A \subseteq X_1$, we can prove that if X is not super- λ , then

$$|T_1| = 1, 1 \leq |S_1| \leq |S_0| \text{ and } S_1 \cup \{1_G\} \leq G.$$

1.2 A is a negative λ -super atom

If $A \subseteq X_0$ and $(1_G, 0) \in A$. Let $A = H \times \{0\}$, then $H \leq G$, by lemma 4.5, $Y = X[A]$ is a regular digraph. We set Y is r_0 regular digraph, then

$$\lambda(X) = |\omega_X^{-1}(A)| = |A|(|S_0| + |T_1| - r_0) \geq \delta(X)(|S_0| + |T_1| - r_0).$$

Since $|T_1| \geq 1$ and $\lambda(X) = \delta(X)$, we have that

$$|A| = \delta(X), |T_1| = 1 \text{ and } |S_0| = r_0.$$

So Y is a $|S_0|$ -regular digraph with order $\delta(X)$, thus

$$|S_0| \leq \delta(X) - 1 \leq \min\{|S_0| + 1, |S_1| + 1\} - 1.$$

So we have that

$$|S_0| \leq |S_1| \text{ and } |S_0| = \delta(X) - 1 \geq 1.$$

So $Y \cong K_{\delta(X)}$ and $H = S_0 \cup \{1_G\} \leq G$.

So if X is not super- λ then

$$|T_1| = 1, 1 \leq |S_0| \leq |S_1| \text{ and } S_0 \cup \{1_G\} \leq G.$$

Similarly, if $A \subseteq X_1$, we can achieve that if X is not super- λ then

$$|T_0| = 1, 1 \leq |S_1| \leq |S_0| \text{ and } S_1 \cup \{1_G\} \leq G.$$

Sufficiency. For condition (1), because of $|T_0| \geq 1$ and $|T_1| \geq 1$, we have

$$\delta(X) = |S_0| + 1.$$

Set $A = S_0 \times \{0\} \cup \{(1_G, 0)\}$, then

$$\min\{\omega^+(A), \omega^-(A)\} = |A| \times \min\{|T_0|, |T_1|\} = |A| = |S_0| + 1 = \delta(X),$$

and because

$$|A| = |S_0 \times \{0\} \cup \{(1_G, 0)\}| \geq 2,$$

so A is a nontrivial λ -fragment.

Condition (2) is similar to condition (1). \square

For the class of \mathcal{L} , by lemma 4.5 and proposition 1.5, we can prove that $Aut(Y)$ acts transitively both on A_i where $Y = X[A]$. Thus if we set $Y' = Y \setminus \{E(Y_0) \cup E(Y_1)\}$ where $Y_i = X[A_i]$, we can easily prove that

$$d_{Y'}^+((g, 0)) = d_{Y'}^-((g, 1)) \text{ and } d_{Y'}^-((g, 0)) = d_{Y'}^+((g, 1)).$$

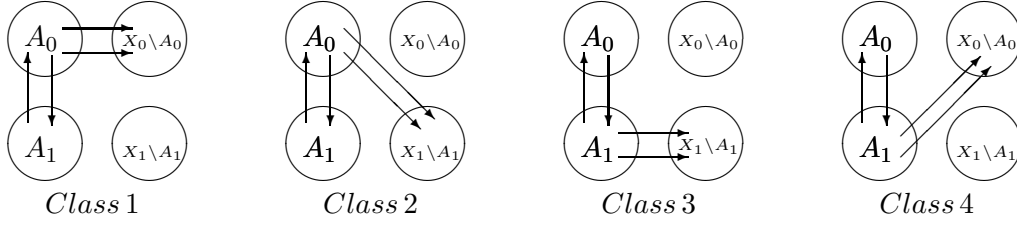
So we set

$$d_{Y'}^+((g, 0)) = d_{Y'}^-((g, 1)) = p \text{ and}$$

$$d_{Y'}^-(g, 0) = d_{Y'}^+(g, 1) = q, \text{ and}$$

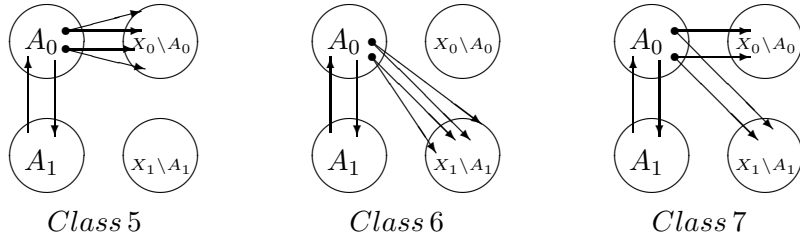
let Y_i is r_i -regular digraph for $i = 0, 1$.

There are some special Classes of digraphs of \mathcal{L} as follows:



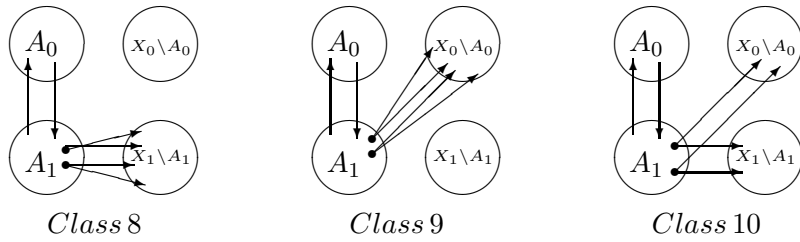
Where A is a λ -superatom and $A_i = X_i \cap A$, and

Class 1 satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and Class 2 satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 1$ and $|T_1| - q = 0$, and Class 3 satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 1$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and Class 4 satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 1$, and all of the above digraphs satisfy that $|A_0| = |A_1| = \delta(X)$.



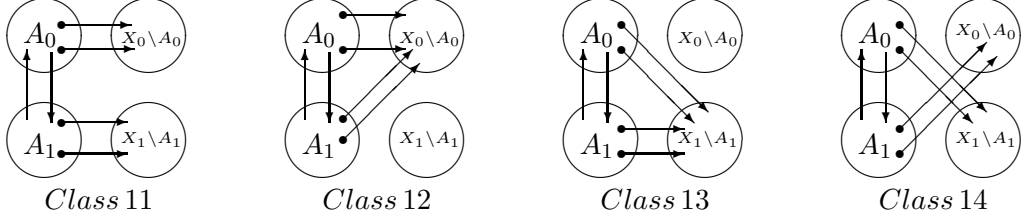
Where A is a λ -superatom and $A_i = X_i \cap A$, and

Class 5 satisfies that $|S_0| - r_0 = 2$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and Class 6 satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 2$ and $|T_1| - q = 0$, and Class 7 satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 0$, $|T_0| - p = 1$ and $|T_1| - q = 0$, and all of the above digraphs satisfy that $|A_0| = |A_1| = \delta(X)/2$.



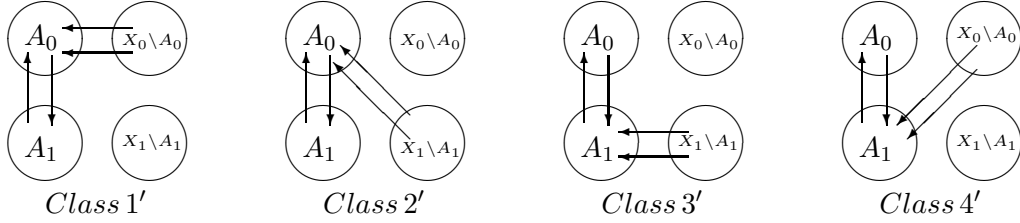
Where A is a λ -superatom and $A_i = X_i \cap A$, and

Class 8 satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 2$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and Class 9 satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 2$, and Class 10 satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 1$, and all of the above digraphs satisfy that $|A_0| = |A_1| = \delta(X)/2$.



Where A is a λ -superatom and $A_i = X_i \cap A$, and

Class 11 satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 1$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and Class 12 satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 0$, $|T_0| - p = 1$ and $|T_1| - q = 1$, and Class 13 satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 1$, $|T_0| - p = 1$ and $|T_1| - q = 0$, and Class 14 satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 1$ and $|T_1| - q = 1$, and all of the above digraphs satisfy that $|A_0| = |A_1| = \delta(X)/2$.



Where A is a λ -superatom and $A_i = X_i \cap A$, and

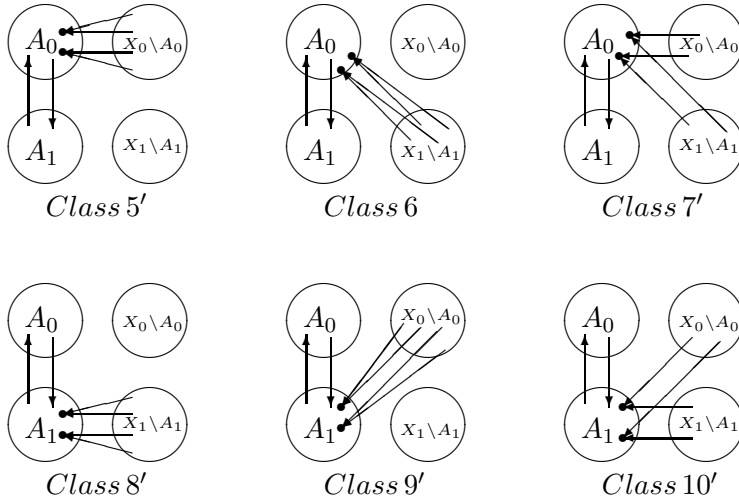
Class 1' satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and Class 2' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 1$, and Class 3' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 1$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and Class 4' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 1$ and $|T_1| - q = 0$, and all of the above digraphs satisfy that $|A_0| = |A_1| = \delta(X)$. Clearly,

Class 1' is equivalent to Class 1,

Class 2' is equivalent to Class 4,

Class 3' is equivalent to Class 3, and

Class 4' is equivalent to Class 2.

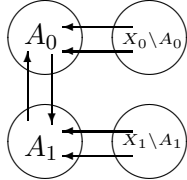


Where A is a λ -superatom and $A_i = X_i \cap A$, and

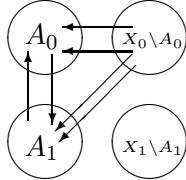
Class 5' satisfies that $|S_0| - r_0 = 2$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and
 Class 6' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 2$, and
 Class 7' satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 0$, $|T_0| - p = 0$ and $|T_1| - q = 1$, and
 Class 8' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 2$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and
 Class 9' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 2$ and $|T_1| - q = 0$, and
 Class 10' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 1$, $|T_0| - p = 1$ and $|T_1| - q = 0$, and
 all of the above digraphs satisfy that $|A_0| = |A_1| = \delta(X)/2$.

Clearly,

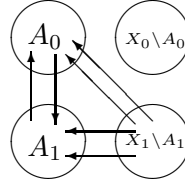
Class 5' is equivalent to Class 5,
 Class 6' is equivalent to Class 9,
 Class 7' is equivalent to Class 12
 Class 8' is equivalent to Class 8
 Class 9' is equivalent to Class 6, and
 Class 10' is equivalent to Class 13.



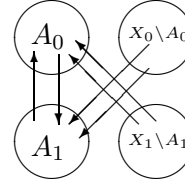
Class 11'



Class 12'



Class 13'



Class 14'

Where A is a λ -superatom and $A_i = X_i \cap A$, and

Class 11' satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 1$, $|T_0| - p = 0$ and $|T_1| - q = 0$, and
 Class 12' satisfies that $|S_0| - r_0 = 1$, $|S_1| - r_1 = 0$, $|T_0| - p = 1$ and $|T_1| - q = 0$, and
 Class 13' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 1$, $|T_0| - p = 0$ and $|T_1| - q = 1$, and
 Class 14' satisfies that $|S_0| - r_0 = 0$, $|S_1| - r_1 = 0$, $|T_0| - p = 1$ and $|T_1| - q = 1$, and
 all of the above digraphs satisfy that $|A_0| = |A_1| = \delta(X)/2$. Clearly,

Class 11' is equivalent to Class 11,
 Class 12' is equivalent to Class 7,
 Class 13' is equivalent to Class 10, and
 Class 14' is equivalent to Class 14.

All of the kinds of the special digraphs of \mathcal{F} are denoted by \mathcal{R} .

Theorem 4.9 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be max- λ , but not super- λ , if X is neither a directed cycle nor a cycle and X doesn't belong to \mathcal{F} , then X belongs to \mathcal{L} if and only if X belongs to \mathcal{R}

Proof. Necessity. Because X is not super- λ and all the λ -superatoms contain at least one vertex of X_0 and X_1 respectively.

2.1 A is a positive λ -super atom.

Then $\delta(X) = \lambda(X) = |\omega_X^+(A)| = |A_0|(|S_0| - r_0 + |T_0| - p) + |A_1|(|S_1| - r_1 + |T_1| - q)$. Since $|A| \geq \delta(X)$ and $|A_0| = |A_1|$, we have $|A_0| = |A_1| \geq \delta(X)/2$. Then if $\lambda(X) = \delta(X) = |\omega_X^+(A)|$ only if one of the following conditions holds.

Case 1 $|S_0| - r_0 + |T_0| - p = 1$ and $|S_1| - r_1 + |T_1| - q = 0$.

Subcase 1.1 $|S_0| - r_0 = 1, |T_0| - p = 0, |S_1| - r_1 = 0$ and $|T_1| - q = 0$, it is Class 1 or Class 1'.

Subcase 1.2 $|S_0| - r_0 = 0, |T_0| - p = 1, |S_1| - r_1 = 0$ and $|T_1| - q = 0$, it is Class 2 or Class 4'.

Case 2 $|S_0| - r_0 + |T_0| - p = 0$ and $|S_1| - r_1 + |T_1| - q = 1$.

Subcase 2.1 $|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 1$ and $|T_1| - q = 0$, it is Class 3 or Class 3'.

Subcase 2.2 $|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 0$ and $|T_1| - q = 1$, it is Class 4 or Class 2'.

Case 3 $|S_0| - r_0 + |T_0| - p = 2$ and $|S_1| - r_1 + |T_1| - q = 0$.

Subcase 3.1 $|S_0| - r_0 = 2, |T_0| - p = 0, |S_1| - r_1 = 0$ and $|T_1| - q = 0$, it is Class 5 or Class 5'.

Subcase 3.2 $|S_0| - r_0 = 0, |T_0| - p = 2, |S_1| - r_1 = 0$ and $|T_1| - q = 0$, it is Class 6 or Class 9'.

Subcase 3.3 $|S_0| - r_0 = 1, |T_0| - p = 1, |S_1| - r_1 = 0$ and $|T_1| - q = 0$, it is Class 7 or Class 12'.

Case 4 $|S_0| - r_0 + |T_0| - p = 0$ and $|S_1| - r_1 + |T_1| - q = 2$.

Subcase 4.1 $|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 2$ and $|T_1| - q = 0$, it is Class 8 or Class 8'.

Subcase 4.2 $|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 0$ and $|T_1| - q = 2$, it is Class 9 or Class 6'.

Subcase 4.3 $|S_0| - r_0 = 0, |T_0| - p = 0, |S_1| - r_1 = 1$ and $|T_1| - q = 1$, it is Class 10 or Class 13'.

Case 5 $|S_0| - r_0 + |T_0| - p = 1$ and $|S_1| - r_1 + |T_1| - q = 1$.

Similarly, we can deduce that under this case, it is Class 11, Class 12, Class 13, Class 14, Class 7', Class 10', Class 11' or Class 14'.

Sufficiency. Clearly. \square

Proposition 4.10 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 1 or Class 1' if and only if

(1) There exists a non-trivial proper subgroup H of G and S_0 contains an element s_0 such that

$$\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H \text{ and } |H| = \delta(X), \text{ and}$$

(2) There is an element $t_0 \in T_0$ such that

$$G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H \text{ and } t_0^{-1} T_0 \subseteq H.$$

Proof. Necessity. Because the class of Class 1 is equivalent to the class of Class 1', without loss of generality, we set X belongs to the class of Class 1, then Assume $(1_G, 0) \in A_0$,

by lemma 4.5, $H_0 \leq G$. Let $H = H_0$, then under this situation we can achieve the following results easily,

(i) $\lambda(X) = |\omega_X^+(A)| = |A_0| = |H_0| = |H| = \delta(X)$, since $|S_0| - r_0 = 1$, $|T_0| - p = 0$, $|S_1| - r_1 = 0$, and $|T_1| - q = 0$,

(ii) $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H_0 = H$, since $|S_0| - r_0 = 1$.

By proposition 4.7, $H_1 = t_0 H_0$ for some $t_0 \in T_0$ and $X_1 = \cup_{i=1}^k (t_0 H_0 g_i) \times \{1\}$,

where $t_0 H_0 g_i \cap t_0 H_0 g_j \neq \emptyset$ if and only if $i = j$ for $1 \leq i, j \leq k$.

Assume that $(1_G, 1) \in (t_0 H_0 g_s) \times \{1\}$, then we can deduce that

$$t_0 H_0 g_s \leq G \text{ and } g_s = h_0^{-1} t_0^{-1}, \text{ where } h_0 \in H_0.$$

Since $|S_1| - r_1 = 0$, we get $G_1 = \langle S_1 \rangle \leq t_0 H_0 g_s = t_0 H_0 h_0^{-1} t_0^{-1} = t_0 H_0 t_0^{-1} = t_0 H t_0^{-1}$.

Since $|T_0| - p = 0$ and $|T_1| - q = 0$, then

$$T_0 H_0 \subseteq H_1 \text{ and } T_1^{-1} H_1 \subseteq H_0,$$

so $T_0 H_0 \subseteq t_0 H_0$ and $T_1^{-1} t_0 H_0 \subseteq H_0$.

It means that $t_0^{-1} T_0 \subseteq H_0 = H$ and $T_1^{-1} t_0 \subseteq H_0 = H$ for some $t_0 \in T_0$.

Sufficiency, set $A = H \times \{0\} \cup (t_0 H) \times \{1\}$,

because $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H$, we can get $|S_0| - r_0 = 1$.

Similarly, because

$$G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H \text{ and } t_0^{-1} T_0 \subseteq H,$$

we can get that

$$|S_1| - r_1 = 0, |T_0| - p = 0 \text{ and } |T_1| - q = 0,$$

so $\lambda(X) = |\omega^+(A)| = |H| = \delta(X)$, and A is not nontrivial. \square

Analogously, we can get the following proposition from 4.11 to 4.23.

Proposition 4.11 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 2 or Class 4' if and only if

(1) There exists a non-trivial proper subgroup H of G such that

$$G_0 = \langle S_0 \rangle \leq H \text{ and } |H| = \delta(X), \text{ and}$$

(2) There are two distinct elements $t_0, t'_0 \in T_0$ such that

$$G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H,$$

$$t'_0 H \cap t_0 H = \emptyset \text{ and } t_0^{-1} (T_0 \setminus \{t'_0\}) \subseteq H. \square$$

Proposition 4.12 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 3 or Class 3' if and only if

(1) There is a non-trivial proper subgroup H of G and some element $s_1 \in S_1$ such that

$$\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq H \text{ and } |H| = \delta(X), \text{ and}$$

(2) There is an element $t_1 \in T_1$ such that

$$G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H \text{ and } t_1 T_1^{-1} \subseteq H. \square$$

Proposition 4.13 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 4 or Class 2' if and only if

(1) There exists a non-trivial proper subgroup H of G such that

- $G_1 = \langle S_1 \rangle \leq H$ and $|H| = \delta(X)$, and
- (2) There are two distinct elements $t_1, t_1' \in T_1$ such that
- $$G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H,$$
- $$t_1'^{-1} H \cap t_1^{-1} H = \emptyset \text{ and } t_1(T_1^{-1} \setminus \{t_1'^{-1}\}) \subseteq H. \quad \square$$

Proposition 4.14 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 5 or Class 5' if and only if

- (1) There exists a non-trivial proper subgroup H of G and S_0 contains two distinct elements s_0, s_0' such that
- $$\langle S_0 \cup \{1_G\} \setminus \{s_0, s_0'\} \rangle \leq H \text{ and } |H| = \delta(X)/2, \text{ and}$$
- (2) There is an element $t_0 \in T_0$ such that
- $$G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H \text{ and } t_0^{-1} T_0 \subseteq H. \quad \square$$

Proposition 4.15 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 6 or Class 9' if and only if

- (1) There exists a non-trivial subgroup H of G such that
- $$G_0 = \langle S_0 \rangle \leq H \text{ and } |H| = \delta(X)/2, \text{ and}$$
- (2) There are three distinct elements $t_0, t_0', t_0'' \in T_0$ such that
- $$G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H, t_0' H \cap t_0 H = \emptyset,$$
- $$t_0'' H \cap t_0 H = \emptyset \text{ and } t_0^{-1}(T_0 \setminus \{t_0', t_0''\}) \subseteq H. \quad \square$$

Proposition 4.16 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 7 or Class 12' if and only if

- (1) There exists a non-trivial proper subgroup H of G , and S_0 contains an element s_0 such that
- $$\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H \text{ and } |H| = \delta(X)/2, \text{ and}$$
- (2) There are two distinct elements $t_0, t_0' \in T_0$ such that
- $$G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, T_1^{-1} t_0 \subseteq H,$$
- $$t_0' H \cap t_0 H = \emptyset \text{ and } t_0^{-1}(T_0 \setminus \{t_0'\}) \subseteq H. \quad \square$$

Proposition 4.17 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected *mixed Cayley digraph*, and X belongs to \mathcal{L} , then X is Class 8 or Class 8' if and only if

- (1) There is a non-trivial subgroup H of G and some $s_1, s_1' \in S_1$ such that
- $$\langle S_1 \cup \{1_G\} \setminus \{s_1, s_1'\} \rangle \leq H \text{ and } |H| = \delta(X)/2, \text{ and}$$
- (2) There is an element $t_1 \in T_1$ such that
- $$G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H \text{ and } t_1 T_1^{-1} \subseteq H. \quad \square$$

Proposition 4.18 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 9 or Class 6' if and only if

- (1) There is a non-trivial proper subgroup H of G such that
- $$G_1 = \langle S_1 \rangle \leq H \text{ and } |H| = \delta(X)/2, \text{ and}$$
- (2) There are three distinct elements $t_1, t_1', t_1'' \in T_1$ such that

$$G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H, t_1^{-1} H \cap t_1'^{-1} H = \emptyset, \\ t_1^{-1} H \cap t_1''^{-1} H = \emptyset \text{ and } t_1(T_1^{-1} \setminus \{t_1'^{-1}, t_1''^{-1}\}) \subseteq H. \quad \square$$

Proposition 4.19 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 10 or Class 13' if and only if

- (1) There is a non-trivial proper subgroup H of G and some element $s_1 \in S_1$ such that $\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
- (2) There are two distinct elements $t_1, t_1' \in T_1$ such that $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, T_0 t_1^{-1} \subseteq H, \\ t_1'^{-1} H \cap t_1^{-1} H = \emptyset \text{ and } t_1(T_1^{-1} \setminus \{t_1'^{-1}\}) \subseteq H. \quad \square$

Proposition 4.20 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 11 or Class 11' if and only if

- (1) There is a non-trivial proper subgroup H of G and S_0 contains an element s_0 such that $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
- (2) There is an element $t_0 \in T_0$ and an element $s_1 \in S_1$ such that $\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq t_0 H t_0^{-1}$ and $T_1^{-1} t_0, t_0^{-1} T_0 \subseteq H. \quad \square$

Proposition 4.21 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 12 or Class 7' if and only if

- (1) There is a non-trivial proper subgroup H of G and S_0 contains an element s_0 such that $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
- (2) There is an element $t_0 \in T_0$ and an element $t_1 \in T_1$ such that $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, t_0^{-1} T_0 \subseteq H, \\ t_1^{-1} t_0 \notin H \text{ and } (T_1^{-1} \setminus \{t_1^{-1}\}) t_0 \subseteq H. \quad \square$

Proposition 4.22 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 13 or Class 10' if and only if

- (1) There is a non-trivial proper subgroup H of G and some element $s_1 \in S_1$ such that $\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
- (2) There is an element $t_1 \in T_1$ and an element $t_0 \in T_0$ such that $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1, t_1 T^{-1} \subseteq H, \\ t_0 t_1^{-1} \notin H \text{ and } (T_0 \setminus \{t_0\}) t_1^{-1} \subseteq H. \quad \square$

Proposition 4.23 $X = MD(G, S_0, S_1, T_0, T_1)$ is a strongly connected mixed Cayley digraph, and X belongs to \mathcal{L} , then X is Class 14 or Class 14' if and only if

- (1) There is a non-trivial proper subgroup H of G such that $G_0 = \langle S_0 \rangle \leq H$ and $|H| = \delta(X)/2$, and
- (2) There are there distinct elements $t_0, t_0' \in T_0, t_1 \in T_1$ such that $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}, t_1^{-1} t_0 \notin H, (T_1^{-1} \setminus \{t_1^{-1}\}) t_0 \subseteq H,$

$$t'_0 H \cap t'_0 H = \emptyset \text{ and } t_0^{-1}(T_0 \setminus \{t'_0\}) \subseteq H. \quad \square$$

From the above discussion, we get the following theorem.

Theorem 4.24 Let $X = MD(G, S_0, S_1, T_0, T_1)$ be $\max-\lambda$, if X is neither a directed cycle nor a cycle and X doesn't belong to F , then X is not $\text{super}-\lambda$ if and only if X satisfies one of the following conditions:

- (1) $|T_0| = 1$ or $|T_1| = 1, 1 \leq |S_0| \leq |S_1|$ and $S_0 \cup \{1_G\} \leq G$.
- (2) $|T_0| = 1$ or $|T_1| = 1, 1 \leq |S_1| \leq |S_0|$ and $S_1 \cup \{1_G\} \leq G$.
- (3) (3.1) There exists a non-trivial proper subgroup H of G and S_0 contains an element s_0 such that
 - $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H$ and $|H| = \delta(X)$, and
 - (3.2) There is an element $t_0 \in T_0$ such that $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}$, $T_1^{-1} t_0 \subseteq H$ and $t_0^{-1} T_0 \subseteq H$.
- (4) (4.1) There exists a non-trivial proper subgroup H of G such that $G_0 = \langle S_0 \rangle \leq H$ and $|H| = \delta(X)$, and
 - (4.2) There are two distinct elements $t_0, t'_0 \in T_0$ such that $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}$, $T_1^{-1} t_0 \subseteq H$, $t'_0 H \cap t_0 H = \emptyset$ and $t_0^{-1}(T_0 \setminus \{t'_0\}) \subseteq H$.
- (5) (5.1) There is a non-trivial proper subgroup H of G and some element $s_1 \in S_1$ such that
 - $\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq H$ and $|H| = \delta(X)$, and
 - (5.2) There is an element $t_1 \in T_1$ such that $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1$, $T_0 t_1^{-1} \subseteq H$ and $t_1 T_1^{-1} \subseteq H$.
- (6) (6.1) There exists a non-trivial proper subgroup H of G such that $G_1 = \langle S_1 \rangle \leq H$ and $|H| = \delta(X)$, and
 - (6.2) There are two distinct elements $t_1, t'_1 \in T_1$ such that $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1$, $T_0 t_1^{-1} \subseteq H$, $t_1'^{-1} H \cap t_1^{-1} H = \emptyset$ and $t_1(T_1^{-1} \setminus \{t_1'^{-1}\}) \subseteq H$.
- (7) (7.1) There exists a non-trivial proper subgroup H of G and S_0 contains two distinct elements s_0, s'_0 such that
 - $\langle S_0 \cup \{1_G\} \setminus \{s_0, s'_0\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
 - (7.2) There is an element $t_0 \in T_0$ such that $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}$, $T_1^{-1} t_0 \subseteq H$ and $t_0^{-1} T_0 \subseteq H$.
- (8) (8.1) There exists a non-trivial subgroup H of G such that $G_0 = \langle S_0 \rangle \leq H$ and $|H| = \delta(X)/2$, and
 - (8.2) There are three distinct elements $t_0, t'_0, t''_0 \in T_0$ such that $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}$, $T_1^{-1} t_0 \subseteq H$, $t'_0 H \cap t_0 H = \emptyset$, $t''_0 H \cap t_0 H = \emptyset$ and $t_0^{-1}(T_0 \setminus \{t'_0, t''_0\}) \subseteq H$.
- (9) (9.1) There exists a non-trivial proper subgroup H of G , and S_0 contains an element s_0 such that
 - $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H$ and $|H| = \delta(X)/2$, and

- (9.2) There are two distinct elements $t_0, t'_0 \in T_0$ such that
 $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}$, $T_1^{-1} t_0 \subseteq H$,
 $t'_0 H \cap t_0 H = \emptyset$ and $t_0^{-1}(T_0 \setminus \{t'_0\}) \subseteq H$.
- (10) (10.1) There is a non-trivial subgroup H of G and some $s_1, s'_1 \in S_1$ such that
 $\langle S_1 \cup \{1_G\} \setminus \{s_1, s'_1\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
(2) There is an element $t_1 \in T_1$ such that
 $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1$, $T_0 t_1^{-1} \subseteq H$ and $t_1 T_1^{-1} \subseteq H$.
- (11) (11.1) There is a non-trivial proper subgroup H of G such that
 $G_1 = \langle S_1 \rangle \leq H$ and $|H| = \delta(X)/2$, and
(11.2) There are there distinct elements $t_1, t'_1, t''_1 \in T_1$ such that
 $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1$, $T_0 t_1^{-1} \subseteq H$, $t_1^{-1} H \cap t'^{-1}_1 H = \emptyset$,
 $t_1^{-1} H \cap t''^{-1}_1 H = \emptyset$ and $t_1(T_1^{-1} \setminus \{t'^{-1}_1, t''^{-1}_1\}) \subseteq H$.
- (12)(12.1) There is a non-trivial proper subgroup H of G and some element $s_1 \in S_1$ such that
 $\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
(12.2) There are two distinct elements $t_1, t'_1 \in T_1$ such that
 $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1$, $T_0 t_1^{-1} \subseteq H$,
 $t'^{-1}_1 H \cap t_1^{-1} H = \emptyset$ and $t_1(T_1^{-1} \setminus \{t'^{-1}_1\}) \subseteq H$.
- (13) (13.1) There is a non-trivial proper subgroup H of G and S_0 contains an element s_0 such that
 $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
(13.2) There is an element $t_0 \in T_0$ and an element $s_1 \in S_1$ such that
 $\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq t_0 H t_0^{-1}$, $T_1^{-1} t_0$ and $t_0^{-1} T_0 \subseteq H$.
- (14) (14.1) There is a non-trivial proper subgroup H of G and S_0 contains an element s_0 such that
 $\langle S_0 \cup \{1_G\} \setminus \{s_0\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
(14.2) There is an element $t_0 \in T_0$ and an element $t_1 \in T_1$ such that
 $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}$, $t_0^{-1} T_0 \subseteq H$,
 $t_1^{-1} t_0 \notin H$ and $(T_1^{-1} \setminus \{t_1^{-1}\}) t_0 \subseteq H$.
- (15) (15.1) There is a non-trivial proper subgroup H of G and some element $s_1 \in S_1$ such that
 $\langle S_1 \cup \{1_G\} \setminus \{s_1\} \rangle \leq H$ and $|H| = \delta(X)/2$, and
(15.2) There is an element $t_1 \in T_1$ and an element $t_0 \in T_0$ such that
 $G_0 = \langle S_0 \rangle \leq t_1^{-1} H t_1$, $t_1 T_1^{-1} \subseteq H$,
 $t_0 t_1^{-1} \notin H$ and $(T_0 \setminus \{t_0\}) t_1^{-1} \subseteq H$.
- (16) (16.1) There is an non-trivial proper subgroup H of G such that
 $G_0 = \langle S_0 \rangle \leq H$ and $|H| = \delta(X)/2$, and
(16.2) There are there distinct elements $t_0, t'_0 \in T_0$, $t_1 \in T_1$ such that
 $G_1 = \langle S_1 \rangle \leq t_0 H t_0^{-1}$, $t_1^{-1} t_0 \notin H$, $(T_1^{-1} \setminus \{t_1^{-1}\}) t_0 \subseteq H$,
 $t'_0 H \cap t_0 H = \emptyset$ and $t_0^{-1}(T_0 \setminus \{t'_0\}) \subseteq H$. \square

So we can conclude that the strongly connected mixed Cayley digraph is $\text{max-}\lambda$ and $\text{super-}\lambda$ but a few exceptions.

References

- [1] J.A. Bondy, U.S.R. Murty. Graph Theory with Applications, North-Holland, New York, 1976.
- [2] J.Y.Chen, Jixiang Meng, Super edge-connectivity of mixed Cayley graph, Science Direct,2007.
- [3] Z.P.Lu, On the Automorphism Groups of Bi-Cayley Graphs, Acta.Sci.Natu. Universitatis Pekinensis, 39(2003)
- [4] M.Mader, Minimale n -fach Kantenzusammenhangenden Graphen, Math. Ann. 191(1971)21-28
- [5] J.X.Meng, Connectivity of vertex and edge transitive graphs, Discrete Appl.Math.127(2003)601-603.
- [6] J.X.Meng, Optimally super-edge-connected transitive graphs,Discrete Math.260(2003)239-248.
- [7] J.X.Meng, Sper-connectivity of Vertex-Transitive Bi-Cayley Graphs and Bipartite Cayley Graph.
- [8] R. Tindell, Connectivity of Cayley digraphs, in:D.Z.Du, D.F.Hsu(Eds.), Combinatorial Network Theory,Klumer,Dordrecht,1996,pp.41-46.
- [9] M.E.Watkins, Connectivity of transitive graphs, J.Comb. Theory 8*1970)23-29.
- [10] M.Y.Xu, Introduction of Finite Group, Vol II,Science Press,Beijing,1999, 384-386.